ASPECTS RELATED TO THE USE OF THE FINITE ELEMENT METHOD FOR MODELLING SOIL BEHAVIOUR UNDER CYCLIC AND DYNAMIC LOADING. APPLICATIONS IN SLOPE STABILITY ANALYSIS

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Rezumat: *Lucrarea prezintă anumite aspecte privind modelarea comportării pământurilor în condiţiile stării plane de deformaţii, prin metoda elementului finit. A fost dezvoltat în acest scop un program de calcul cu sursă deschisă, care utilizează elemente de formă dreptunghiulară, cu 8 grade de libertate şi care poate fi obţinut de la adresa http://matgts.sourceforge.net. Se prezintă formularea unora dintre modelele constitutive utilizate, precum şi anumite probleme legate de implementarea lor. Variaţia deplasărilor se consideră liniară în interiorul fiecărui element şi se propune o metodă simplă pentru evitarea supraestimării rigidităţii la încovoiere a elementelor, care ar cauza imposibilitatea modelării cedării la forfecare a materialului. Sunt de asemenea prezentate, interpretate şi comparate cu rezultate obţinute prin alte metode mai simple, rezultatele câtorva analize neliniare, statice şi dinamice, ale unor taluzuri sub acţiuni seismice, în urma cărora s-au obţinut deplasări şi acceleraţii critice.*

Abstract*: The paper presents certain aspects of modelling soil behaviour under plane strain conditions, by means of the finite element method. An open source computational program was developed for this purpose, which implements rectangular shell elements with 8 degrees of freedom and can be downloaded from http://matgts.sourceforge.net. The formulation of some of the constitutive models used is presented, as well as specific problems related to their implementation. The displacements inside each element are assumed to have linear variations and a simple method is proposed for the elimination of volumetric locking (which would cause overestimation of the bending stiffness of the elements, leading to unacceptable results, especially after shear failure of the material), without increasing the number of equations. The results of several nonlinear static and dynamic analyses of slopes under seismic loading, in which displacements and critical accelerations were calculated, are shown, interpreted and compared to results obtained through other, simpler methods.*

Keywords: plasticity, rectangular, dynamic, shear, free, open, source

1. Introduction

Several types of computational methods are currently used for the analysis of slopes under seismic loading. Most of these methods use limit equilibrium [1-3] or finite elements [4,5]. Considering the rapid advancements in computer technology, the discrete element method may also start to be widely used at some point in the future. The objective of this paper is to study certain aspects related to the use of the finite element method.

2. Theoretical aspects

2.1. Constitutive behaviour of soils

Generally, Equation (1) is used for the mathematical expression of the stress-strain behaviour of material models:

$$\{\dot{\sigma}\}=[D]\cdot\{\dot{\epsilon}\} \tag{1}$$

Where $\{\dot{\sigma}\}=(\dot{\sigma}_x\ \dot{\sigma}_y\ \dot{\sigma}_z\ \dot{\tau}_{yz}\ \dot{\tau}_{xz}\ \dot{\tau}_{xy})^T$, $\{\dot{\epsilon}\}=(\dot{\epsilon}_x\ \dot{\epsilon}_y\ \dot{\epsilon}_z\ \dot{\gamma}_{yz}\ \dot{\gamma}_{xz}\ \dot{\gamma}_{xy})^T$ (stresses and strains)

Because soils have nonlinear behaviours, it is necessary that the constitutive equations relate increments of stress and strain, as indicated above and that matrix [D] depends on the current and past stresses. For an isotropic linear elastic material, matrix [D] is given by Equation (2).

In order to apply these relations to a real geotechnical problem, certain assumptions and idealisations must be made, including simplification of the geometry or of the boundary conditions. Problems like the analysis of retaining walls, continuous footings and slopes generally have one dimension very large in comparison with the other two. If the forces and boundary conditions are perpendicular to and independent of that dimension, all cross sections will be the same and the case of plane strain, defined by Equations (3), can be considered.

$$[D]=\frac{E}{(1-2\nu)(1+\nu)}\cdot\begin{vmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2}\end{vmatrix} \quad (2)$$

$$\epsilon_z=0;\gamma_{yz}=0;\gamma_{xz}=0 \quad (3)$$

The case of plane strain will be detailed below. In these conditions, the constitutive behaviour of an isotropic linear elastic material is expressed by Equations (4) and (5):

$$\begin{vmatrix}\dot{\sigma}_x\\ \dot{\sigma}_y\\ \dot{\tau}_{xy}\end{vmatrix}=\frac{E}{(1-2\nu)(1+\nu)}\cdot\begin{vmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{vmatrix}\cdot\begin{vmatrix}\dot{\epsilon}_x\\ \dot{\epsilon}_y\\ \dot{\gamma}_{xy}\end{vmatrix} \quad (4)$$

$$\sigma_z=\nu\cdot(\sigma_x+\sigma_y) \quad (5)$$

Equation (5), in which E is the elasticity modulus and ν is Poisson's ratio, will be assumed for all material models described below, so that only the stresses σ_x , σ_y and τ_{xy} have to be considered in the analysis. This is acceptable if the Mohr-Coulomb failure condition is used and the principal intermediate stress, σ_2 , is assumed to be equal to σ_z .

2.2. Mohr-Coulomb model

The Mohr-Coulomb model is an elastic - perfectly plastic constitutive model, which uses a yield function given by Equations (6) and (7):

$$F(\sigma)=(-\sigma_3)-(-\sigma_1)\cdot K_p-2\cdot c\cdot\sqrt{K_p} \quad (6)$$

$$K_p=tg^2(45^0+\frac{\phi}{2}) \quad (7)$$

where σ_1 and σ_3 are principal stresses (Figure 1), ϕ is the internal friction angle and c is the cohesion.

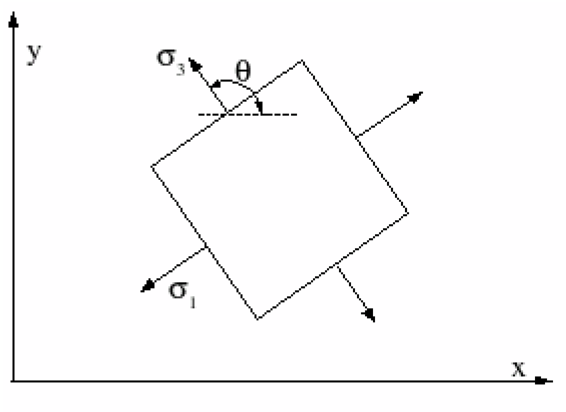


Fig. 1 - Principal stresses

When F(σ) < 0, the material is considered to have an elastic behaviour, expressed in Equation (4).

When F(σ) = 0, the behaviour of the material is plastic and another relation must be determined, having the form shown in Equation (8):

$$\{\dot{\sigma}\}=[B]\cdot\{\dot{\epsilon}\} \tag{8}$$

where $\{\dot{\sigma}\}=(\dot{\sigma}_x\ \ \dot{\sigma}_y\ \ \dot{\tau}_{xy})^T$, $\{\dot{\epsilon}\}=(\dot{\epsilon}_x\ \ \dot{\epsilon}_y\ \ \dot{\gamma}_{xy})^T$ (stresses and strains in plane xOy)

Because the yield function is expressed in terms of principal stresses, a constitutive matrix, [A], also expressed in terms of principal stresses, will be determined first, and then matrix [B] will be determined through its rotation, as shown in Equations (9):

$$\begin{pmatrix}\dot{\sigma}_3\\ \dot{\sigma}_1\end{pmatrix}=\begin{pmatrix}a_{1,1} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{pmatrix}\cdot\begin{pmatrix}\dot{\epsilon}_{\sigma 3}\\ \dot{\epsilon}_{\sigma 1}\end{pmatrix};\quad [A]=\begin{pmatrix}a_{1,1} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{pmatrix};$$

$$[B]=[C]^T\cdot[A]\cdot[C];\quad [C]=\begin{pmatrix}\cos^2\theta & \sin^2\theta & \sin\theta\cdot\cos\theta\\ \sin^2\theta & \cos^2\theta & -\sin\theta\cdot\cos\theta\end{pmatrix};$$

$$\sigma_x>\sigma_y\rightarrow\theta=\frac{arctg\left(\frac{2\cdot\tau_{xy}}{\sigma_x-\sigma_y}\right)}{2}+\frac{\pi}{2};\quad \sigma_x\leqslant\sigma_y\rightarrow\theta=\frac{\pi+arctg\left(\frac{2\cdot\tau_{xy}}{\sigma_x-\sigma_y}\right)}{2}+\frac{\pi}{2} \tag{9}$$

where θ is the angle shown in Figure 1 and $\varepsilon_{\sigma1}$ and $\varepsilon_{\sigma2}$ are strains in the directions of the principal stresses, σ_1 and σ_3 .

Relations (10) ... (14) show how matrix [A] is determined. Relation (13) is applicable when the shear strain increment is 0 and $\dot{\epsilon}_1=\dot{\epsilon}_3$, like in Equation (11).

$$\dot{F}=\frac{\delta F}{\delta\sigma_1}\cdot\dot{\sigma}_1+\frac{\delta F}{\delta\sigma_3}\cdot\dot{\sigma}_3=0\rightarrow\dot{\sigma}_3=K_p\cdot\dot{\sigma}_1 \tag{10}$$

$$\begin{pmatrix}K_p\cdot\dot{\sigma}_1\\ \dot{\sigma}_1\end{pmatrix}=\begin{pmatrix}a_{1,1} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{pmatrix}\cdot\begin{pmatrix}0.5\\ 0.5\end{pmatrix} \tag{11}$$

$$\alpha \cdot \begin{pmatrix} K_p \cdot \dot{\sigma}_1 \\ \dot{\sigma}_1 \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \cdot \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix} \tag{12}$$

$$p = \frac{\dot{\sigma}_1 + \dot{\sigma}_2 + \dot{\sigma}_3}{3} = \frac{(1+\nu)\cdot(\dot{\sigma}_1 + \dot{\sigma}_3)}{3} = \frac{(1+\nu)\cdot(1+K_p)\cdot\dot{\sigma}_1}{3} = K \cdot \epsilon_v = K \cdot (\epsilon_1 + \epsilon_3) = K \cdot 1 \tag{13}$$

$$[A] = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \frac{6 \cdot K}{(1+\nu)\cdot(1+K_p)} \cdot \begin{pmatrix} 0.5 K_p \cdot (1+\alpha) & 0.5 K_p \cdot (1-\alpha) \\ 0.5 \cdot (1+\alpha) & 0.5 \cdot (1-\alpha) \end{pmatrix} \tag{14}$$

where $\dot{\sigma}_1$ and $\dot{\sigma}_3 = K_p \cdot \dot{\sigma}_1$ are principal stress increments caused by a volumetric strain increment equal to 1, α is a dilatancy coefficient equal to the ratio between plastic volumetric strain and plastic shear strain and K is the bulk modulus of the material.

When $F(\sigma) = 0$, the behaviour of the material model for the next load increment can be described using the constitutive matrix [B] in equations (9) - plastic behaviour, as well as the constitutive matrix in equation (4) - elastic behaviour. The matrix which leads to the lowest value of the elastic potential energy of the element is chosen.

The elastic potential energy of the material per unit of volume is given by equation (15):

$$w_p = 0.5 \cdot \{\sigma\}^T \cdot \{\epsilon_{elastic}\} = 0.5 \cdot \{\sigma\}^T \cdot [B]^{-1} \cdot \{\sigma\} \tag{15}$$

where [B] is the elastic constitutive matrix from Equation (4).

2.3. Finite element analysis

For the purpose of using constitutive models such as the one described at 1.2, a computational program (currently available at http://matgts.sourceforge.net) was developed, in which the soil is modelled by rectangular finite shell elements with 4 nodes and 8 degrees of freedom, as illustrated in Figure 2. The stiffnesses of such an element can be calculated using Equation (16). The strains in each element are considered to have linear variations.

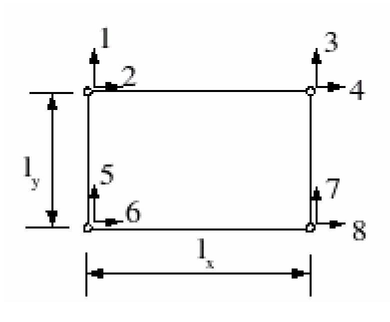


Fig. 2 - Finite element type

$$k_{j,i} = \int_0^{lx} \left(\int_0^{ly} \left(\sigma_x^{(i)} \cdot \epsilon_x^{(j)} + \sigma_y^{(i)} \cdot \epsilon_y^{(j)} + \tau_{xy}^{(i)} \cdot \gamma_{xy}^{(j)} \right) dy \right) dx \tag{16}$$

where $\sigma_x^{(i)}$, $\sigma_y^{(i)}$, $\tau_{xy}^{(i)}$ are stresses caused by a displacement equal to 1, applied in the direction of the degree of freedom *i* and $\epsilon_x^{(j)}$, $\epsilon_y^{(j)}$, $\gamma_{xy}^{(j)}$ are strains caused by a displacement equal to 1, applied in the direction of the degree of freedom *j*.

Equations (16) overestimate the bending stiffness of the element and can lead to unrealistic results, especially when shear failure occurs. The plastic constitutive matrix for a material with $\alpha = 0$, calculated according to (9) ... (14), has the form shown in Equation (17). In that case, the element, which has no shear stiffness, should also have no bending stiffness. This is solved by calculating the stiffnesses of such an element using Equations (18), which were determined directly from the condition that the bending stiffness is 0, instead of Equations (16).

$$[B]=\begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix}=\begin{pmatrix} a & a & 0 \\ b & b & 0 \\ c & c & 0 \end{pmatrix} \tag{17}$$

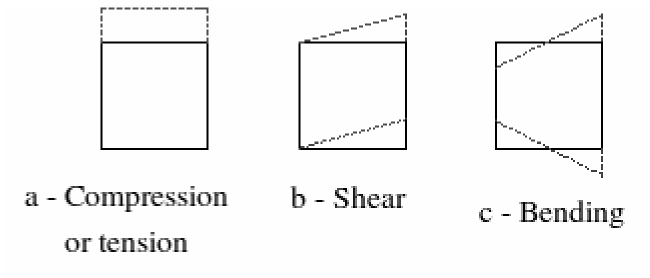


Fig. 3 - Deformations of an element

$$\begin{aligned}
k_{1,1}&=k_{1,3}=k_{7,5}=k_{7,7}=-k_{1,5}=-k_{1,7}=-k_{7,1}=-k_{7,3}=\frac{b}{4}\cdot\frac{l_x}{l_y}-\frac{c}{4}\\
k_{2,2}&=k_{2,6}=k_{8,4}=k_{8,8}=-k_{2,4}=-k_{2,8}=-k_{8,2}=-k_{8,6}=\frac{a}{4}\cdot\frac{l_y}{l_x}-\frac{c}{4}\\
k_{3,1}&=k_{3,3}=k_{5,5}=k_{5,7}=-k_{3,5}=-k_{3,7}=-k_{5,1}=-k_{5,3}=\frac{b}{4}\cdot\frac{l_x}{l_y}+\frac{c}{4}\\
k_{4,4}&=k_{4,8}=k_{6,2}=k_{6,6}=-k_{4,2}=-k_{4,6}=-k_{6,4}=-k_{6,8}=\frac{a}{4}\cdot\frac{l_y}{l_x}+\frac{c}{4}\\
k_{1,2}&=k_{1,6}=k_{7,4}=k_{7,8}=-k_{1,4}=-k_{1,8}=-k_{7,2}=-k_{7,6}=\frac{c}{4}\cdot\frac{l_y}{l_x}-\frac{b}{4}\\
k_{3,4}&=k_{3,8}=k_{5,2}=k_{5,6}=-k_{3,2}=-k_{3,6}=-k_{5,4}=-k_{5,8}=\frac{c}{4}\cdot\frac{l_y}{l_x}+\frac{b}{4}\\
k_{2,1}&=k_{2,3}=k_{8,5}=k_{8,7}=-k_{2,5}=-k_{2,7}=-k_{8,1}=-k_{8,3}=\frac{c}{4}\cdot\frac{l_x}{l_y}-\frac{a}{4}\\
k_{4,1}&=k_{4,3}=k_{6,5}=k_{6,7}=-k_{4,5}=-k_{4,7}=-k_{6,1}=-k_{6,3}=\frac{c}{4}\cdot\frac{l_x}{l_y}+\frac{a}{4}
\end{aligned} \tag{18}$$

However, equations (18) lead to numerical instability, so the stiffnesses of an element whose constitutive matrix is of the form shown in (17) are be calculated in the program with (19):

$$k_{j,i}=p\cdot k_{j,i}^{(1)}+(1-p)\cdot k_{j,i}^{(2)} \tag{19}$$

where $k_{j,i}^{(1)}$ are stiffnesses calculated with (16), $k_{j,i}^{(2)}$ are stiffnesses calculated with (18) and p = 0,01.

In general, the constitutive matrix [B] is decomposed, as shown in (20) and the stiffnesses of all elements are calculated as follows: a stiffness matrix is calculated considering for the element's material the matrix $[B]_1$ and using equations (19), another stiffness matrix is calculated using $[B]_2$ and equations (16) and the 2 stiffness matrices are added. The resulting matrix is the total stiffness matrix of the element.

$$[B]=[B]_1+[B]_2=\begin{pmatrix} \frac{b_{1,1}+b_{1,2}}{2} & \frac{b_{1,1}+b_{1,2}}{2} & 0 \\ \frac{b_{2,1}+b_{2,2}}{2} & \frac{b_{2,1}+b_{2,2}}{2} & 0 \\ \frac{b_{3,1}+b_{3,2}}{2} & \frac{b_{3,1}+b_{3,2}}{2} & 0 \end{pmatrix}+\begin{pmatrix} \frac{b_{1,1}-b_{1,2}}{2} & \frac{b_{1,2}-b_{1,1}}{2} & b_{1,3} \\ \frac{b_{2,1}-b_{2,2}}{2} & \frac{b_{2,2}-b_{2,1}}{2} & b_{2,3} \\ \frac{b_{3,1}-b_{3,2}}{2} & \frac{b_{3,2}-b_{3,1}}{2} & b_{3,3} \end{pmatrix} \tag{20}$$

To allow for the effect of pore water pressures, the program mentioned above can use 2 separate stiffness matrices for each element, thus allowing effective stresses and pore water pressures to be calculated independently (assuming undrained conditions).

3. Analysis examples

The following analyses were performed for the model in Figure 4:

- pseudo-static analyses using Bishop's method of slices (1955) and the finite element method, to determine the critical horizontal acceleration;
- dynamic analyses using Newmark's sliding block method (1965) and the finite element method, for the N-S component of the seismic motion recorded at INCERC, in Bucharest, during the earthquake from March 4, 1977.

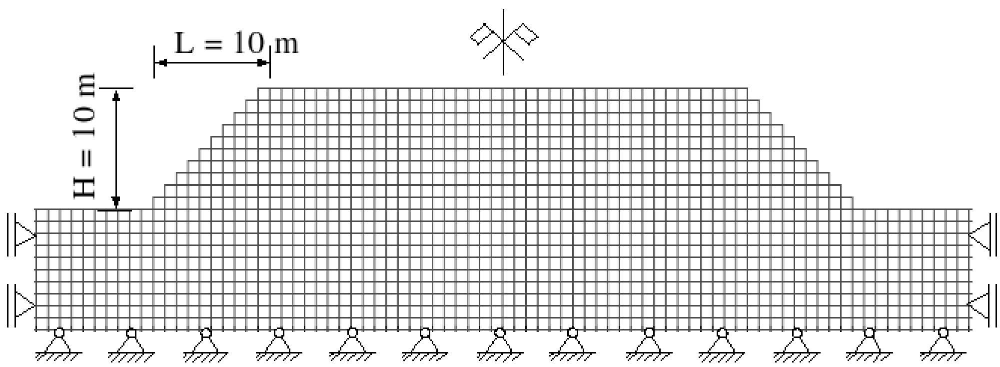


Fig. 4 - Model used in analyses

The Mohr-Coulomb model described above was used in the finite element analyses and the following parameters were considered: γ = 20 KN/m^3, ϕ = 10^0, c = 30 KPa, E = 660000 KPa, ν = 0.45, α = 0. In the pseudo-static analyses, the seismic forces were replaced with static horizontal forces, applied at the centres of the slices in the case of Bishop's method and at the nodes in the case of the finite element method. Each horizontal force was equal to the corresponding weight multiplied by a coefficient k_H. The critical value of the coefficient was k_H = 0.15 for Bishop's method and $k_H \simeq 0.17$ for the finite element method, the results of which are illustrated in Figure 6. The accelerogram used for the dynamic analyses is shown in Figure 5.

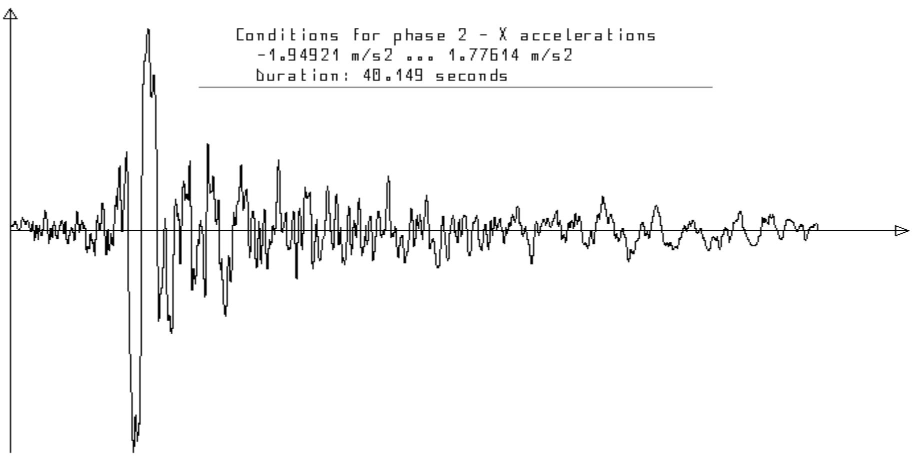


Fig. 5 - accelerogram used in the dynamic analyses

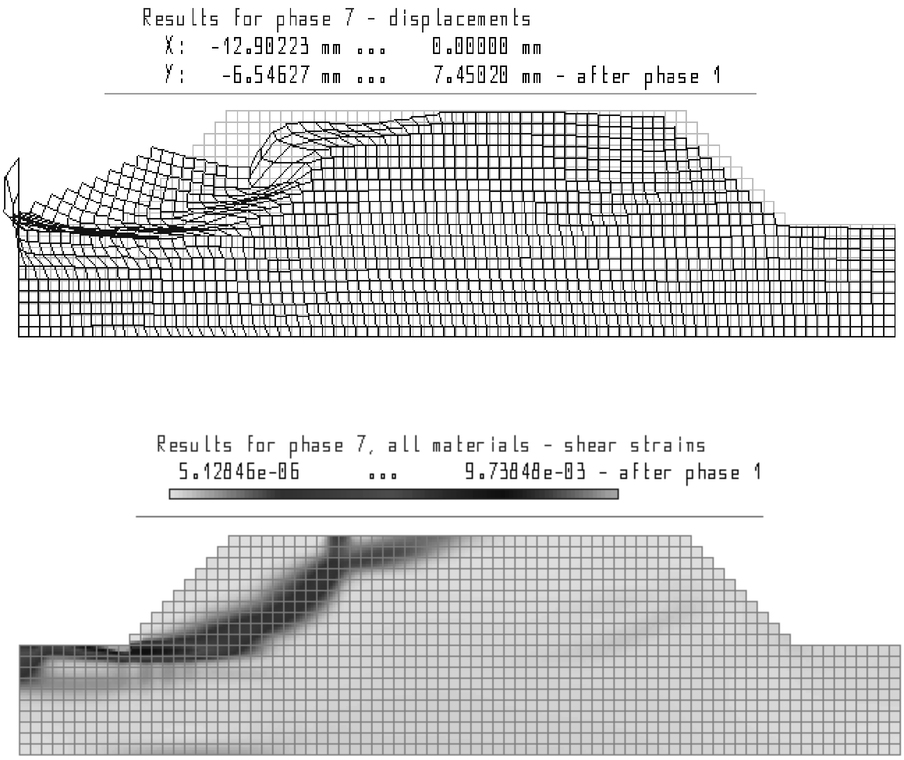
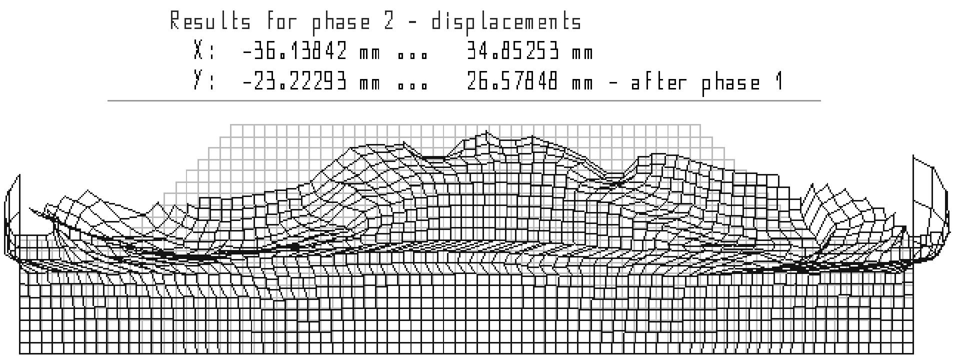


Fig. 6 - Displacements and shear strains calculated for kH = 0.17



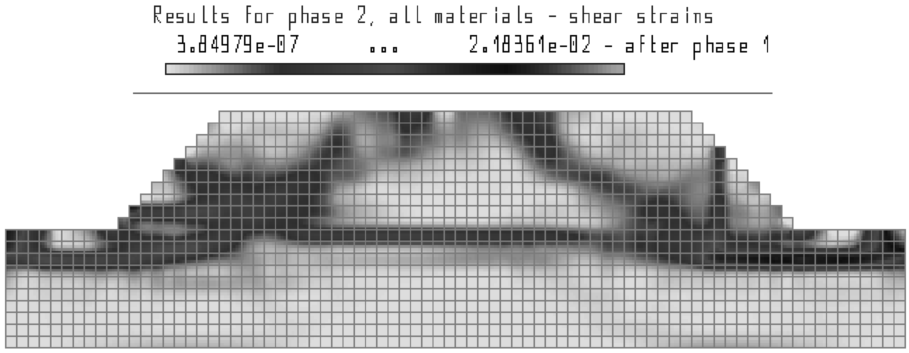


Fig. 7 - Displacements and shear strains resulted from the dynamic analysis

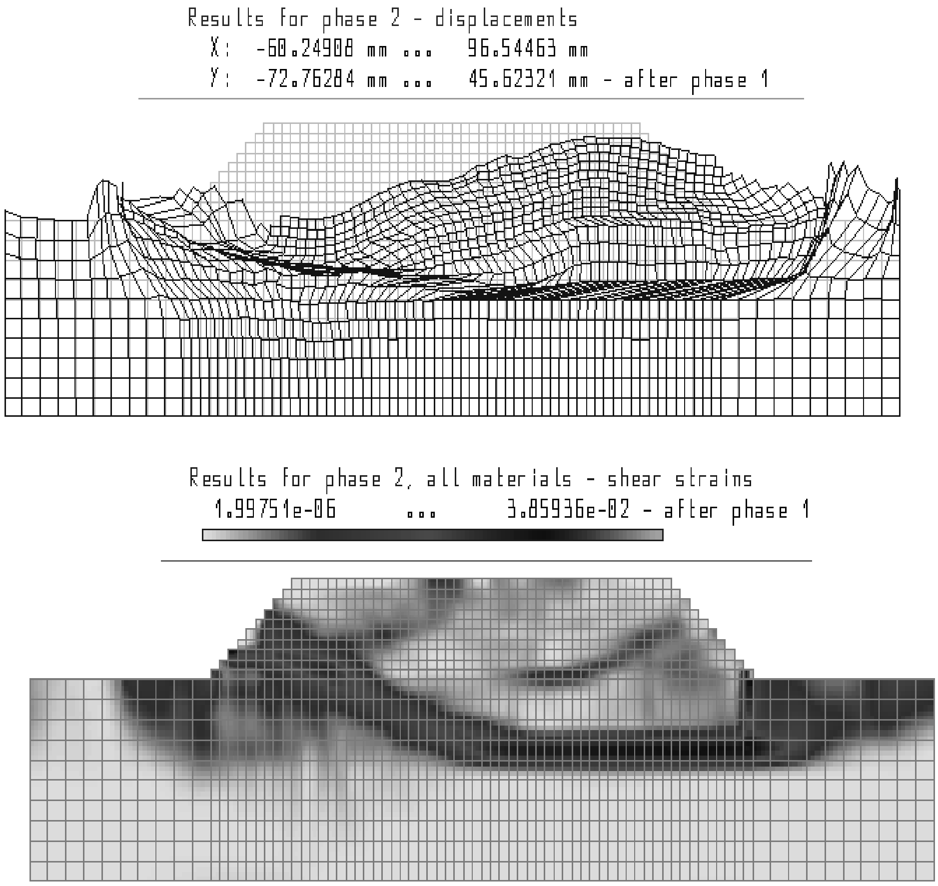


Fig. 8 - Displacements and shear strains resulted from the dynamic analysis (different boundary conditions)

Two sets of boundary conditions were considered in the dynamic analyses, as shown in Figures 7 and 8. The permanent displacement calculated for the same accelerogram with Newmark's sliding block method and considering a critical horizontal acceleration $a_{cr,H} = k_{cr,H} \cdot g = 0.15 \cdot g$ (the value obtained using Bishop's method) was approximately 30 mm, comparable to the values shown in Figures 7 and 8.

4. Conclusions

In this case, the displacements calculated with Newmark's sliding block method were comparable to those resulted from the finite element analysis. However, Newmark's method only provides a value for the horizontal displacement, it gives no information about how the model could deform and ignores a large number of factors (deformability, dynamic amplification, pore water pressures etc.). Although several commercial finite element analysis computer programs designed for modelling soil currently exist, the development and use of a free such program was preferred, since commercial software most often comes with restrictive licensing conditions and technical measures to prevent its use and is therefore not considered a viable option by the author of this paper.

References

[1] Bishop A. - The Use of the Slip Circle in the Stability Analysis of Slopes. Geotechnique 5, 1955

[2] Newmark N. - Effects of Earthquakes on Dams and Embankments, Geotechnique 15, 1965

[3] Spencer E. - Earth Slopes Subjected to Lateral Acceleration. Journal of Geotechnical Engineering 104, 1978

[4] Potts David M, Zdravković Lidija. - Finite Element Analysis in Geotechnical Engineering. Thomas Telford Publishing, London, 1999

[5] Schanz T., Vermeer P.A., Bonnier P.G. The hardening soil model: Formulation and verification. Beyond 2000 in Computational Geotechnics - 10 Years of PLAXIS, 1999