Solving equations of motion

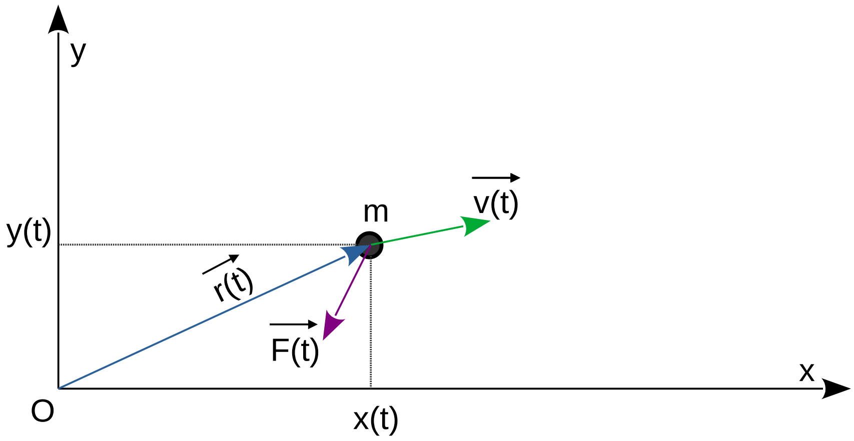

Figure 1 - particle moving in system xOy

In Figure 1:

t - time;
m - mass of particle;
r(t) = (x(t) y(t))^T - position vector;
v(t) = d**r**(t) / dt - velocity;
F(t) - force acting on particle.

According to some guy named Isaac:

$$\vec{a}(t)=\frac{d\vec{v}(t)}{dt}=\frac{d}{dt}\left(\frac{d\vec{r}(t)}{dt}\right)=\frac{d^2\vec{r}(t)}{dt^2}=\ddot{\vec{r}}(t)=\frac{\vec{F}(t)}{m} \tag{1}$$

where **a**(t) is the acceleration of the particle.

The force **F** can also depend on other parameters besides t, like the velocity of the particle, in the case of a cannonball moving through air, for instance. Equation (1) can be solved numerically, which usually means finding **r**(t) and **v**(t) at t_1, t_2, t_3,...,t_i, t_{i+1},..., where $t_{i+1} = t_i + \Delta t$. The initial values **r**(t_0) and **v**(t_0) are considered to be known. In general:

$$\vec{r}(t_{i+1})=\vec{r}(t_i)+\int_{t_i}^{t_{i+1}}\vec{v}(t)\,dt \quad ; \quad for\ t_i\leqslant t_j\leqslant t_{i+1}\rightarrow\vec{v}(t_j)=\vec{v}(t_i)+\int_{t_i}^{t_j}\vec{a}(t)\,dt \tag{2}$$

If **F**(t) = 0, then **a**(t) = 0, **v**(t) = **v**(t_i) (the velocity is constant) and:

$$\vec{r}(t_{i+1})=\vec{r}(t_i)+\vec{v}(t_i)\cdot\Delta t\rightarrow\begin{pmatrix}x_{i+1}\\y_{i+1}\end{pmatrix}=\begin{pmatrix}x_i\\y_i\end{pmatrix}+\begin{pmatrix}v_x(t_i)\\v_y(t_i)\end{pmatrix}\cdot\Delta t \tag{3}$$

Equation (3) is known as *Euler's method* and can also be used when **F**(t) ≠ 0, in which case the velocity is only approximated as constant between t_i and t_{i+1}. If the force is considered constant in a time step, i.e. **F**(t) = **F**(t_i) for $t_i \le t < t_{i+1}$, then **a**(t) = **a**(t_i) and:

$$\vec{v}(t_j)=\vec{v}(t_i)+\vec{a}(t_i)\cdot(t_j-t_i) \quad ; \tag{4}$$

$$\vec{r}(t_{i+1}) = \vec{r}(t_i) + \vec{v}(t_i)\cdot\Delta t + \vec{a}(t_i)\cdot\frac{(\Delta t)^2}{2} \tag{5}$$

Equation (5) is known as *the SUVAT method*. If the acceleration is considered to have a linear variation, then:

$$t_i \leqslant t < t_{i+1} \rightarrow \vec{a}(t) = \frac{d^2\vec{r}(t)}{dt^2} = \ddot{\vec{r}}(t) = \vec{b}\cdot(t - t_i) + \vec{c} \tag{6}$$

where **b** and **c** are constant vectors:

$$\vec{b} = \dddot{\vec{r}}(t_i) = \frac{d^3\vec{r}(t_i)}{dt^3} \simeq \frac{\ddot{\vec{r}}(t_{i+1}) - \ddot{\vec{r}}(t_i)}{(t_{i+1} - t_i)} \quad ; \quad \vec{c} = \ddot{\vec{r}}(t_i) \tag{7}$$

$$\vec{r}(t_{i+1}) = \vec{r}(t_i) + \dot{\vec{r}}(t_i)\cdot\Delta t + \ddot{\vec{r}}(t_i)\cdot\frac{(\Delta t)^2}{2} + \dddot{\vec{r}}(t_i)\cdot\frac{(\Delta t)^3}{6} \tag{8}$$

In general, if the first n derivatives of **r**(t) can be determined:

$$\vec{r}(t_{i+1}) = \sum_{k=0}^{n} \frac{d^{(k)}\vec{r}(t_i)}{dt^{(k)}}\cdot\frac{(\Delta t)^k}{k!} \tag{9}$$

Equation (9) is called *the nth Taylor polynomial of* ***r****(t)*.

Verlet's method, also commonly used and suitable for computing trajectories of planets and other particles moving in various force fields, uses a slightly different approach:

$$\ddot{\vec{r}}(t_i) = \vec{a}(t_i) = \frac{\dfrac{\vec{r}(t_{i+1}) - \vec{r}(t_i)}{\Delta t} - \dfrac{\vec{r}(t_i) - \vec{r}(t_{i-1})}{\Delta t}}{\Delta t} = \frac{\vec{r}(t_{i+1}) - 2\cdot\vec{r}(t_i) + \vec{r}(t_{i-1})}{(\Delta t)^2} \tag{10}$$

$$\vec{r}(t_{i+1}) = 2\cdot\vec{r}(t_i) - \vec{r}(t_{i-1}) + a(t_i)\cdot(\Delta t)^2 \tag{11}$$

Since **r**(t_{i+1}) depends on **r**(t_{i-1}), Equation (11) can not be used to calculate **r**(t_1), but one of the other methods can be used for the first step.